

# A Generalization of the Power Law Distribution with Nonlinear Exponent

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## Abstract

The power law distribution is usually used to fit data in the upper tail of the distribution. However, commonly it is not valid to model data in all the range. In this paper, we present a new family of distributions, the so-called Generalized Power Law (GPL), which can be useful for modeling data in all the range and possess power law tails. To do that, we model the exponent of the power law using a nonlinear function which depends on data and two parameters. Then, we provide some basic properties and some specific models of that new family of distributions. After that, we study a relevant model of the family, with special emphasis on the quantile and hazard functions, and the corresponding estimation and testing methods. Finally, as an empirical evidence, we study how the debt is distributed across municipalities in Spain. We check that power law model is only valid in the upper tail; we show analytically and graphically the competence of the new model with municipal debt data in the whole range; and we compare the new distribution with other well-known distributions including the Lognormal, the Generalized Pareto, the Fisk, the Burr type XII and the Dagum models.

**Key Words:** Power law behavior; Whole range fitting; Complexity; Municipal debt

## 1 Introduction

Many empirical analysis of diverse real phenomena (the population of the cities, the annual income of the people, the solar flare intensity, the

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failures in power grids, the protein interaction degree, etc) have confirmed the power law behavior in the upper tail of their distributions - the largest values of the variable of interest, above a certain lower bound, can be modeled statistically by a classical Pareto distribution, with shape parameter  $\alpha$  also known as exponent of the power law or simply constant  $\alpha$  (see, for example, [1, 2, 3, 4, 5]). That empirical evidence comes with many advantages: it can help us to understand the underlying data generating process [6]; it gives us tools for computer simulation of those phenomena [7]; etc.

However, Pareto distribution is not usually valid to model those real phenomena in the whole range - if we consider high, medium and low ranges of those data all together, the power law behaviour usually disappears. For example: failures in power grids can be described by the Lomax distribution [8, 9]; or in the case of protein interaction networks of three species (*C.elegans*, *S.cerevisiae* and *E.coli*), or in the case of the metabolic networks with human and yeast data, the lognormal distribution provides the best description for the empirical data [10], in the whole range.

The Pareto distribution hierarchy, composed by Pareto type I (Power Law), Pareto type II (with Lomax distribution as a special case), Pareto type III and type IV, is a well known extension of the power law [11, 12]. Those family of distributions, also known as Generalized Pareto distributions, have extended the scope of the classical Pareto model, as for example, with the failures in power grids and the Lomax distribution, as mentioned previously.

The aim of this study is twofold. Firstly, to explore the properties of a new family of GPL distributions that we could use to model real phenomena in the whole range, phenomena with power law tail. Secondly, to provide empirical evidence of the efficacy of those distributions with real datasets. Our primary hypothesis was that Pareto shape parameter, the exponent  $\alpha$ , is not constant and varies according to a non-linear function  $g$  which depends on data [13]. We found a surprisingly rich family of distributions, with only three parameters, which includes Pareto and Pareto Positive Stable (PPS) distributions as special cases, and we also found that a new distribution, a relevant model of that family, is a good alternative for modeling debt data of the indebted municipalities in Spain in the whole range.

The rest of this paper is organized as follows: in Section 2, we introduce a new family of GPL distributions; in Section 3, we present a new distribution, which belongs to that new family; an empirical application of that new distribution to municipal debt with Spanish data is included in Section 4; finally, the conclusions are given in Section 5.

## 2 A new family of Generalized Power Law distributions

In this section we obtain the new family of Generalized Power Law (GPL) distributions. Our idea is to construct an extension of the Power law, where the exponent is not constant and is modeled by using a non-linear function of the data. Then, let consider a real function  $g : (1, \infty) \rightarrow \mathbb{R}^+$  continuous, positive and differentiable on  $(1, \infty)$  satisfying the following conditions,

$$\lim_{z \rightarrow 1^+} z^{g(z)} = 1 \text{ and } \lim_{z \rightarrow \infty} z^{g(z)} = \infty, \quad (1)$$

and

$$\frac{g'(z)}{g(z)} > \frac{-1}{z \log(z)}, \quad \forall z > 1. \quad (2)$$

Now, using  $g(\cdot)$ , we define the function

$$F(x) = 1 - \left(\frac{x}{\sigma}\right)^{-g(x/\sigma)}, \quad x > \sigma, \quad (3)$$

and  $F(x) = 0$  if  $x \leq \sigma$ . Note that  $\sigma$  is a scale parameter. We have the following Theorem.

**Theorem 1** *Let consider the functional form defined in (3), where the function  $g(\cdot)$  satisfies conditions (1) and (2). Then (3) is a genuine cumulative distribution function (cdf).*

**Proof:** It is direct to check that  $F(-\infty) = 0$ ,  $F(\infty) = 1$  and  $F(x)$  is right continuous. Finally,  $F(x)$  is nondecreasing since,

$$g\left(\frac{x}{\sigma}\right) + \frac{x}{\sigma} \log\left(\frac{x}{\sigma}\right) g'\left(\frac{x}{\sigma}\right) > 0, \quad \forall x > \sigma,$$

using condition (2). ■

The family of distributions define in Eq.(3) includes the classical Pareto distribution (also known as Power Law or Pareto type I distribution) [11, 14] as a special case when  $g(z) = \alpha$ ,  $\forall z > 1$  with  $\alpha > 0$ .

## 2.1 Basic Properties

The survival function  $S(x) = \Pr(X > x) = 1 - F(x)$  is given by:

$$S(x) = \left(\frac{x}{\sigma}\right)^{-g(x/\sigma)}, \quad x > \sigma,$$

and  $S(x) = 1$  if  $x \leq \sigma$ .

The probability density function (pdf) of that family of distributions is given by,

$$f(x) = \frac{dF(x)}{dx} = \left\{ g\left(\frac{x}{\sigma}\right) + \left(\frac{x}{\sigma}\right) \log\left(\frac{x}{\sigma}\right) g'\left(\frac{x}{\sigma}\right) \right\} \frac{S(x)}{x},$$

if  $x > \sigma$  and  $f(x) = 0$  if  $x \leq \sigma$ .

The hazard function  $h(x) = \frac{f(x)}{\Pr(X > x)} = \frac{f(x)}{S(x)}$  is as follows:

$$h(x) = \frac{g\left(\frac{x}{\sigma}\right) + \left(\frac{x}{\sigma}\right) \log\left(\frac{x}{\sigma}\right) g'\left(\frac{x}{\sigma}\right)}{x},$$

if  $x > \sigma$  and  $h(x) = 0$  if  $x \leq \sigma$ . Some graphics of this family are included in Section 3 for a relevant special case.

## 2.2 Some models of generalized power law and extensions

In this section we present some specific models of Generalized Power Law distributions and we also provide some extensions of that new family of distributions. To model the  $g(\cdot)$  function, we choose some flexible functions which depend on two parameters  $\alpha$  and  $\beta$ , and that include as special case the constant function by setting  $\beta = 0$ .

Table 1 provides some models of Generalized Power Law distributions, where we have reported the  $g(z)$  function, the survival function and the pdf. The simplest choice, that is,  $g(z) = \alpha$ , corresponds to the usual power law, or classical Pareto distribution. The choice  $g(z) = \alpha \log^\beta(z)$  corresponds to the PPS distribution [13, 15]. As far as we know, the rest of models are new.

On the other hand, we can consider some extensions of these models. These extensions can be obtained using the Pareto types II or IV models [12, 16, 17], instead of the usual classical Pareto distribution. In these extensions, we incorporate a new location parameter or new location and shape

**Table 1**

Some examples of distributions which belongs to the new family of distributions described.

$g(z)$	$\beta$	$S(x)$	$f(x)$
$\alpha$		$\left(\frac{x}{\sigma}\right)^{-\alpha}$	$\alpha \frac{S(x)}{x}$
$\alpha \log^\beta(z)$	$\beta > -1$	$\left(\frac{x}{\sigma}\right)^{-\alpha \log^\beta(x/\sigma)}$	$\alpha(\beta + 1) \log^\beta(x/\sigma) \frac{S(x)}{x}$
$\alpha z^\beta$	$\beta \geq 0$	$\left(\frac{x}{\sigma}\right)^{-\alpha(x/\sigma)^\beta}$	$\alpha[1 + \beta \log(x/\sigma)] \left(\frac{x}{\sigma}\right)^\beta \frac{S(x)}{x}$
$\alpha + \beta \log z$	$\beta \geq 0$	$\left(\frac{x}{\sigma}\right)^{-\alpha - \beta \log(x/\sigma)}$	$[\alpha + 2\beta \log(x/\sigma)] \frac{S(x)}{x}$
$\alpha + \beta z$	$\beta \geq 0$	$\left(\frac{x}{\sigma}\right)^{-\alpha - \beta(x/\sigma)}$	$[\alpha + \beta(x/\sigma)(1 + \log(x/\sigma))] \frac{S(x)}{x}$
$\alpha - \beta \left(\frac{z-1}{z \log z}\right)$	$\beta \leq \alpha$	$\left(\frac{x}{\sigma}\right)^{-\alpha + \beta \left[\frac{(x/\sigma)-1}{\log(x/\sigma)}\right]}$	$\left[\alpha - \frac{\beta}{(x/\sigma)}\right] \frac{S(x)}{x}$
$\alpha - \frac{\beta}{z}$	$\beta \leq \alpha$	$\left(\frac{x}{\sigma}\right)^{-\alpha + \beta \sigma/x}$	$\left[\alpha + \frac{\beta}{(x/\sigma)}(\log(x/\sigma) - 1)\right] \frac{S(x)}{x}$
$\alpha + \beta \left(\frac{z-1}{\log(z)}\right)$	$\beta \geq 0$	$\left(\frac{x}{\sigma}\right)^{-\alpha - \beta \left[\frac{(x/\sigma)-1}{\log(x/\sigma)}\right]}$	$[\alpha + \beta(x/\sigma)] \frac{S(x)}{x}$
$\alpha + \beta \left(\frac{\log z}{1 + \log z}\right)$	$\beta \geq -\alpha$	$\left(\frac{x}{\sigma}\right)^{-\alpha - \beta \left[\frac{\log(x/\sigma)}{\log(x/\sigma)+1}\right]}$	$\left[\alpha + \beta \frac{\log(x/\sigma)[\log(x/\sigma) + 2]}{[\log(x/\sigma) + 1]^2}\right] \frac{S(x)}{x}$
$\alpha + \beta \left(\frac{z-1}{z}\right)$	$\beta \geq -\alpha$	$\left(\frac{x}{\sigma}\right)^{-\alpha - \beta \left[\frac{(x/\sigma)-1}{(x/\sigma)}\right]}$	$\left[\alpha + \beta \frac{\log(x/\sigma) - 1 + (x/\sigma)}{(x/\sigma)}\right] \frac{S(x)}{x}$
$\alpha \left(\frac{\log z}{1 + \log z}\right)^\beta$	$\beta > -1$	$\left(\frac{x}{\sigma}\right)^{-\alpha \left[\frac{\log(x/\sigma)}{\log(x/\sigma)+1}\right]^\beta}$	$\alpha \left[\frac{\log(x/\sigma) + 1 + \beta}{\log(x/\sigma) + 1}\right] \left[\frac{\log(x/\sigma)}{\log(x/\sigma) + 1}\right]^\beta \frac{S(x)}{x}$
$\alpha \left(\frac{z-1}{z}\right)^\beta$	$\beta \geq -1$	$\left(\frac{x}{\sigma}\right)^{-\alpha \left[\frac{(x/\sigma)-1}{(x/\sigma)}\right]^\beta}$	$\alpha \left[\frac{(x/\sigma) - 1 + \beta \log(x/\sigma)}{(x/\sigma) - 1}\right] \left[\frac{(x/\sigma) - 1}{(x/\sigma)}\right]^\beta \frac{S(x)}{x}$

parameters, respectively, and the support of the distribution is  $(\mu, \infty)$ , where  $\mu \geq 0$ . In this situation, the  $g(\cdot)$  function is continuous, positive and differentiable on the interval  $(0, \infty)$  and satisfy:

$$\lim_{z \rightarrow 0^+} (1+z)^{g(z)} = 1 \text{ and } \lim_{z \rightarrow \infty} (1+z)^{g(z)} = \infty, \quad (4)$$

and

$$\frac{g'(z)}{g(z)} > \frac{-1}{(1+z) \log(1+z)}, \quad \forall z > 0. \quad (5)$$

In the case of Pareto II distribution, the new family of distributions is defined in terms of the cdf,

$$F(x; \mu, \sigma) = 1 - \left[ 1 + \left( \frac{x - \mu}{\sigma} \right) \right]^{-g\{(x-\mu)/\sigma\}}, \quad x > \mu,$$

and  $F(x) = 0$  if  $x \leq \mu$ , where  $\mu$  is a location parameter,  $\sigma > 0$  is a scale parameter and  $g(\cdot)$  satisfies conditions (4) and (5).

In the case of the Pareto IV distribution, the new family of distributions is given by,

$$F(x; \mu, \sigma, \gamma) = 1 - \left[ 1 + \left( \frac{x - \mu}{\sigma} \right)^{(1/\gamma)} \right]^{-g\{[(x-\mu)/\sigma]^{1/\gamma}\}}, \quad x > \mu,$$

and  $F(x) = 0$  if  $x \leq \mu$ , where  $\mu$  is a location parameter,  $\sigma > 0$  is a scale parameter,  $\gamma > 0$  a shape parameter and  $g(\cdot)$  satisfies again conditions (4),(5).

### 3 A relevant model

In this section we study a relevant model of Generalized Power Law distribution. This model corresponds to the choice  $g(z) = \alpha \left( \frac{\log z}{1 + \log z} \right)^\beta$  in Table 1, and the cdf is given by,

$$F(x; \alpha, \beta, \sigma) = 1 - \exp \left\{ -\alpha \frac{[\log(x/\sigma)]^{\beta+1}}{[\log(x/\sigma) + 1]^\beta} \right\}, \quad x \geq \sigma, \quad (6)$$

and  $F(x) = 0$  if  $x < \sigma$ , where  $\alpha > 0$  and  $\beta > -1$  are shape parameters, and  $\sigma > 0$  is a scale parameter. A random variable with cdf given by Eq.(6) will

be denoted by  $X \sim \mathcal{GPL}(\alpha, \beta, \sigma)$ . This family includes the classical Pareto distribution (Power Law) when  $\beta = 0$ . We have  $\mathcal{GPL}(\alpha, 0, \sigma) \equiv \mathcal{Pa}(\alpha, \sigma)$ .

The concept of tail equivalent (see [18, 19, 20, 21, 22, 23, 24]) is satisfied by the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution. The following Theorem shows that the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution exhibit a power law behaviour when  $x$  is large.

**Theorem 2** *The  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution, defined in Eq.(6), and the Pareto distribution are right tail equivalent*

**Proof:** The proof is direct and it is based on the fact that,

$$\lim_{x \rightarrow \infty} \frac{1 - F(x)}{1 - G(x)} = \lim_{x \rightarrow \infty} \frac{\exp \left\{ -\alpha \frac{[\log(x/\sigma)]^{\beta+1}}{[\log(x/\sigma)+1]^\beta} \right\}}{(x/\sigma)^{-\alpha}} = 1,$$

where  $G(x)$  is the cumulative distribution function of the Pareto distribution.

■

In the following Theorem we show the domain of attraction for maxima (see [2, 25, 26, 27, 28, 29, 30, 31])

**Theorem 3** *The  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution belongs to the Maximum Domain of Attraction of the Fréchet distribution  $\mathcal{GPL}(\alpha, \beta, \sigma) \in MDA(\Phi_\alpha)$*

**Proof:** We must check that  $1 - F(x)$  is of regular variation of index  $-\alpha$ ,

$$\lim_{x \rightarrow \infty} \frac{1 - F(tx)}{1 - F(x)} = t^{-\alpha}, \forall t > 0,$$

or in other words,  $1 - F(x)$  can be expressed as  $L(x)x^{-\alpha}$  where  $L(x)$  is a slowly varying function ( $\lim_{x \rightarrow \infty} L(tx)/L(x) = 1$ , for any  $t > 0$ ):

$$1 - F(x) = (x/\sigma)^{-\alpha} \left\{ \left[ \frac{\log(x/\sigma)}{\log(x/\sigma)+1} \right]^\beta - 1 \right\} (x/\sigma)^{-\alpha} \sim L(x)x^{-\alpha},$$

which means that  $\mathcal{GPL}(\alpha, \beta, \sigma)$  is a heavy-tailed distribution and, for that, it can be useful for statistical modeling of phenomena with extremely large observations. ■

### 3.1 Basic Properties

The survival function  $S(x) = \Pr(X > x) = 1 - F(x)$  is given by:

$$S(x) = \exp \left\{ -\alpha \frac{[\log(x/\sigma)]^{\beta+1}}{[\log(x/\sigma) + 1]^\beta} \right\}, \quad x \geq \sigma, \quad (7)$$

and  $S(x) = 1$  if  $x < \sigma$ . Figure 1 shows the survival function  $S(x)$  of the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution, given by Eq.(7), for different values of the shape parameters  $\alpha$  and  $\beta$ , in log-log scale.

The pdf of the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution is given by,

$$f(x) = \frac{\alpha}{x} \left[ \frac{\log(x/\sigma) + 1 + \beta}{\log(x/\sigma) + 1} \right] \left[ \frac{\log(x/\sigma)}{\log(x/\sigma) + 1} \right]^\beta \exp \left\{ -\alpha \frac{[\log(x/\sigma)]^{\beta+1}}{[\log(x/\sigma) + 1]^\beta} \right\}, \quad x \geq \sigma \quad (8)$$

and  $f(x) = 0$  if  $x < \sigma$ . Figure 2 shows the probability density function  $f(x)$ , given by Eq.(8), for zero-modal and uni-modal curves. Remark that  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution, as a distribution of the MDA of the Fréchet distribution, satisfies the von Mises condition [32]:

$$\lim_{x \rightarrow \infty} \frac{xf(x)}{S(x)} = \lim_{x \rightarrow \infty} \alpha \left[ \frac{\log(x/\sigma) + 1 + \beta}{\log(x/\sigma) + 1} \right] \left[ \frac{\log(x/\sigma)}{\log(x/\sigma) + 1} \right]^\beta = \alpha > 0.$$

The hazard function is given by (see also Section 2.1):

$$h(x) = \frac{f(x)}{\Pr(X > x)} = \frac{f(x)}{S(x)} = \frac{\alpha}{x} \left[ \frac{\log(x/\sigma) + 1 + \beta}{\log(x/\sigma) + 1} \right] \left[ \frac{\log(x/\sigma)}{\log(x/\sigma) + 1} \right]^\beta, \quad x \geq \sigma,$$

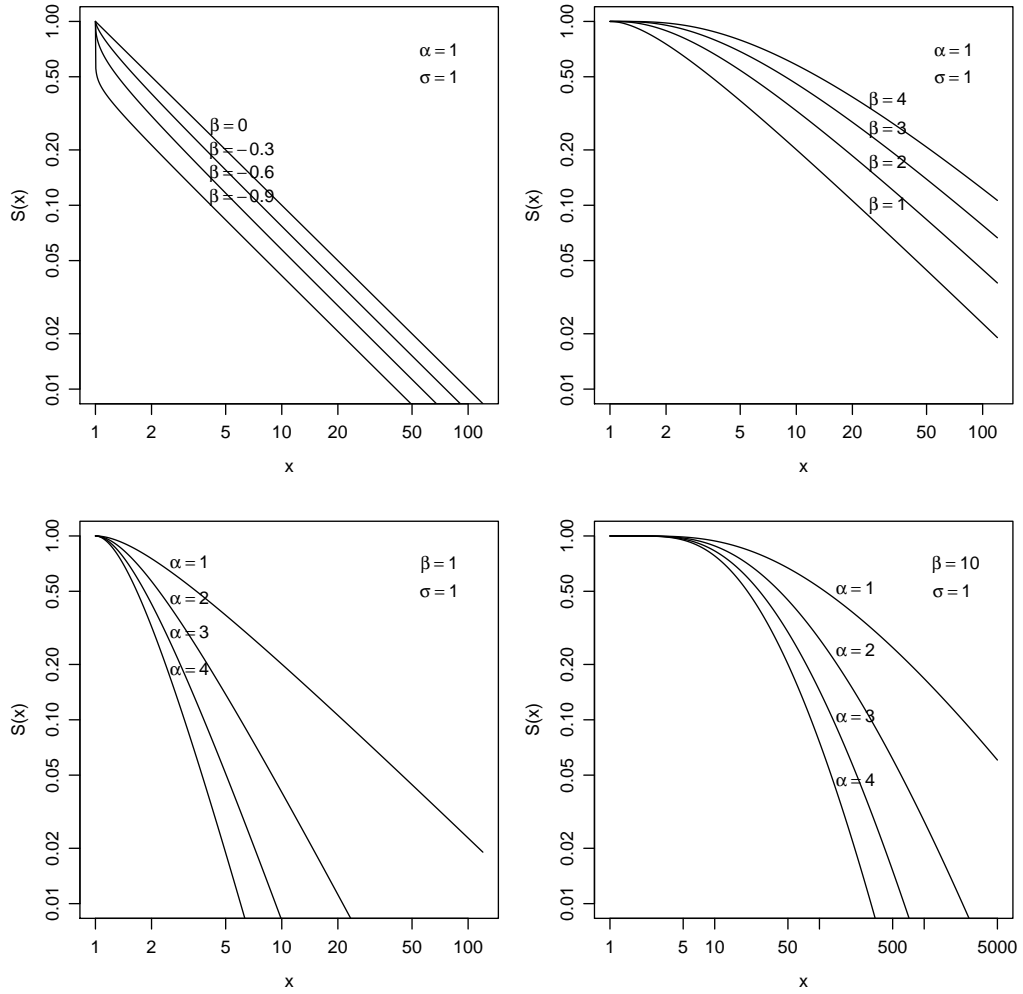
and  $h(x) = 0$  if  $x < \sigma$ . See Figure 3 for different shapes.

The quantile function  $Q(p) = F^{-1}(p)$  is defined implicitly as follows,

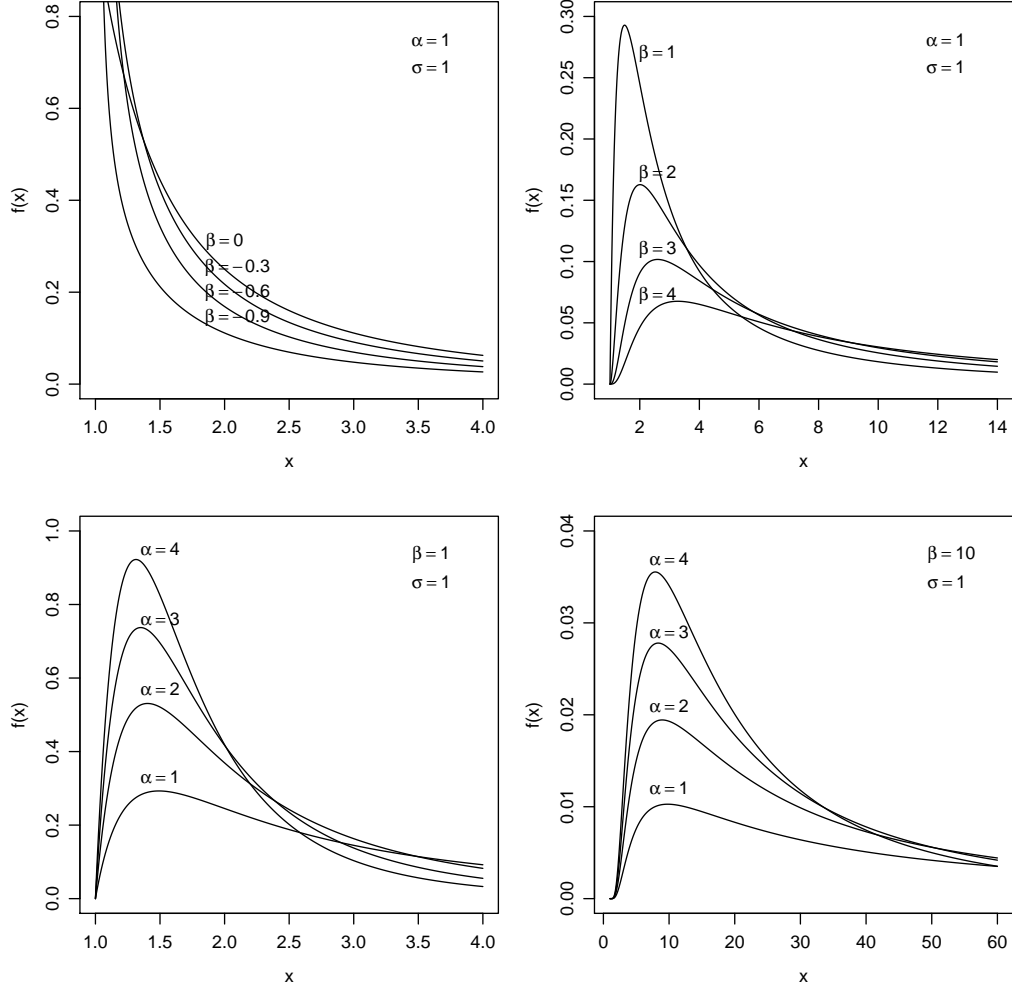
$$\frac{-\log(1-p) (\log [Q(p)/\sigma] + 1)^\beta}{(\log [Q(p)/\sigma])^{\beta+1}} = \alpha, \quad 0 < p < 1,$$

which can be used to simulate the random variable  $X \sim \mathcal{GPL}(\alpha, \beta, \sigma)$  of our interest by the inverse transform method, from a random variable uniformly distributed  $U \sim \mathcal{U}(0, 1)$  and  $X = F^{-1}(U)$  [33, 34].

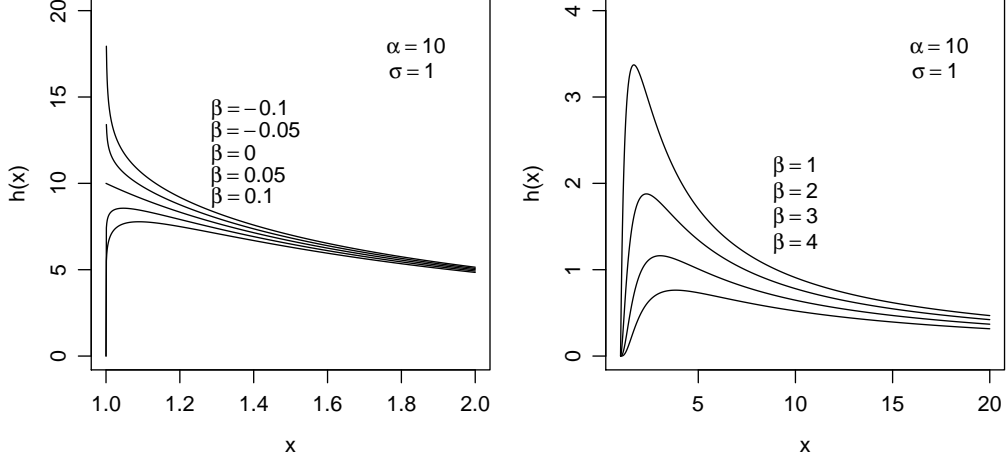




**Figure 1:** Plots of the survival function of the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution with  $\sigma = 1$  and: (up-left)  $\alpha = 1$  and  $\beta = -0.9, -0.6, -0.3, 0$ ; (up-right)  $\alpha = 1$  and  $\beta = 1, 2, 3, 4$ ; (down-left)  $\beta = 1$  and  $\alpha = 1, 2, 3, 4$ ; (down-right)  $\beta = 10$  and  $\alpha = 1, 2, 3, 4$



**Figure 2:** Plots of the probability density function of the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution with  $\sigma = 1$  and: (up-left)  $\alpha = 1$  and  $\beta = -0.9, -0.6, -0.3, 0$ ; (up-right)  $\alpha = 1$  and  $\beta = 1, 2, 3, 4$ ; (down-left)  $\beta = 1$  and  $\alpha = 1, 2, 3, 4$ ; (down-right)  $\beta = 10$  and  $\alpha = 1, 2, 3, 4$



**Figure 3:** Plots of the hazard function of the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution with  $\sigma = 1$  and: (left)  $\alpha = 10$  and  $\beta = -0.1, -0.05, 0, 0.05, 0.1$ ; (right)  $\alpha = 10$  and  $\beta = 1, 2, 3, 4$

### 3.2 Estimation and Testing

The  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution can be fitted using the method of maximum likelihood [35]. Let  $x_1, \dots, x_n$  be a sample of size  $n$  drawn from a  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution. The log-likelihood function can be expressed as follows,

$$\begin{aligned} \log \ell(\alpha, \beta, \sigma) &= \sum_{i=1}^n \log f(x_i) = n \log(\alpha) - \sum_{i=1}^n \log(x_i) \\ &\quad + \sum_{i=1}^n \log[\log(x_i/\sigma) + 1 + \beta] + \beta \sum_{i=1}^n \log[\log(x_i/\sigma)] \\ &\quad - (\beta + 1) \sum_{i=1}^n \log[\log(x_i/\sigma) + 1] - \alpha \sum_{i=1}^n \frac{\log^{\beta+1}(x_i/\sigma)}{[\log(x_i/\sigma) + 1]^\beta}, \end{aligned} \quad (9)$$

where  $(\alpha, \beta, \sigma)$  is the unknown parameter vector of the model,  $f(x)$  is the pdf of the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution defined in Eq.(8), and the maximum likelihood estimation of the parameter vector  $(\hat{\alpha}, \hat{\beta}, \hat{\sigma})$  is the one that maximizes the likelihood function  $\log \ell(\alpha, \beta, \sigma)$ .

The normal equations can be obtained by taking partial derivatives of

Eq.(9) with respect to  $\alpha, \beta, \sigma$ , and equating them to zero. If we use the auxiliary random variable  $W = \log(X/\sigma)$ , and we represent its observed values by  $w_i = \log(x_i/\sigma), i = 1, \dots, n$ , the normal equations can be expressed as:

$$\begin{aligned} \frac{\partial \log \ell}{\partial \alpha} = 0 &\Rightarrow \alpha = n \left[ \sum_{i=1}^n \frac{w_i^{\beta+1}}{(w_i + 1)^\beta} \right]^{-1}, \\ \frac{\partial \log \ell}{\partial \beta} = 0 &\Rightarrow \sum_{i=1}^n \frac{1}{w_i + 1 + \beta} + \log \left( \frac{w_i}{w_i + 1} \right) \left[ 1 - \frac{\alpha w_i^{\beta+1}}{(w_i + 1)^\beta} \right] = 0, \\ \frac{\partial \log \ell}{\partial \sigma} = 0 &\Rightarrow \sum_{i=1}^n \frac{\beta(\beta + 1)[\log(w_i) + 1]^\beta - \alpha w_i \log^{\beta+1}(w_i)[\log(w_i) + 1 + \beta]^2}{w_i \log(w_i)[\log(w_i) + 1]^{\beta+1}[\log(w_i) + 1 + \beta]} = 0. \end{aligned} \quad (10)$$

The previous equations (10) can be solved by numerical methods. In this study, maximum likelihood estimates of the parameters  $\alpha$ ,  $\beta$  and  $\sigma$  were computed by using the R software function `optimx` [36, 37], with the limited memory quasi-Newton L-BFGS-B algorithm (in which bounds constraints are permitted) [38, 39, 40] - for that, we took  $\sigma_0$  equal to half of the smallest value of the sample, as the initial value of  $\sigma$ , and  $\alpha_0, \beta_0$  the values obtained from the first two partial derivatives (in Eq.(10)) just plugging  $\sigma_0$  into them.

We can compare the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution with other different models by using two model selection criteria: the Akaike information criterion (*AIC*), defined by [41],

$$AIC = -2 \log L + 2d;$$

or the Bayesian information criterion (*BIC*), defined by [42]

$$BIC = \log L - \frac{1}{2}d \log n; \quad (11)$$

where  $\log L = \log \ell(\hat{\alpha}, \hat{\beta}, \hat{\sigma})$  is the log-likelihood (see Eq. 9) of the model evaluated at the maximum likelihood estimates,  $d$  is the number of parameters (in the case of the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution,  $d = 3$ ) and  $n$  is the number of data. The model chosen is the one with the smallest value of *AIC* statistic or with the largest value of *BIC* statistic.

We can use rank-size plots (on a log-log scale) for graphical model validation. We can plot the complementary of the theoretical cdf (multiplied by  $n + 1$ ) of the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  model together with the scatter plot of the points

(observed data)  $\log(rank_i)$  versus  $\log(x_{(i)})$ ,  $i = 1, \dots, n$ , where  $x_{(1)} \leq \dots \leq x_{(i)} \leq \dots \leq x_{(n)}$  is the ordered sample of  $X$  and  $rank_i = n + 1 - i$  [8].

Finally, we can test the goodness-of-fit of the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  model by a Kolmogorov-Smirnov ( $KS$ ) test method based on bootstrap resampling [2, 8, 43, 44, 45, 46, 47] as follows: (1) calculating the empirical  $KS$  statistic of the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  model for the observed data,  $KS = \sup |F_n(x_i) - F(x_i; \hat{\alpha}, \hat{\beta}, \hat{\sigma})|$ ,  $i = 1, 2, \dots, n$ , where  $F(x_i; \hat{\alpha}, \hat{\beta}, \hat{\sigma})$  is the theoretical cdf of the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  model fitted by maximum likelihood, in a sample value, and  $F_n(x_i) \approx (n + 1)^{-1} \sum_{j=1}^n I_{[x_j \leq x_i]}$  is the empirical cdf in a sample value with the indicated plotting position formula [48]; (2) generate, by simulation, enough  $\mathcal{GPL}(\alpha, \beta, \sigma)$  synthetic data sets (in this study, we generated 10000 data sets), with the same sample size  $n$  - notice that the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  quantile function  $Q(p) = F^{-1}(p)$  is defined implicitly, then, for this study, we used the R software function `uniroot` [36]; (3) fit each  $\mathcal{GPL}(\alpha, \beta, \sigma)$  synthetic data set by maximum likelihood and obtained its theoretical cdf; (4) calculate the  $KS$  statistic for each  $\mathcal{GPL}(\alpha, \beta, \sigma)$  synthetic data set - with its own theoretical cdf; (5) calculate the  $p$ -value as the fraction of  $\mathcal{GPL}(\alpha, \beta, \sigma)$  synthetic data sets with a  $KS$  statistic greater than the empirical  $KS$  statistic; (6) null hypothesis  $H_0$ : *the data follow the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  model* can be rejected with the 0.1 level of significance if  $p\text{-value} < 0.1$ .

## 4 Empirical application to municipal debt in Spain

In this section, as an illustration, we show that  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution can be useful for modeling Spanish municipalities debt.

### 4.1 The data

We considered debt data of the indebted municipalities in Spain. There are three levels of government in Spain: the State, the Autonomous Communities and the Local Entities [49, 50]. Municipalities belong to the third one - as a reference, there were 8117 municipalities in Spain in 2014 [51]. The expenditure of those councils, directed at providing essential local services to their citizens (street cleaning, local police, etc.), is financed through different sources: transfers, local taxes, public fares, etc. For several reasons (infrastructure investment, etc.), they can decide to contract debt - taking into

account the municipal debt control of the institutional borrowing restrictions [52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62]. Our data sets were composed of information of Spanish indebted municipalities, whose debt was at least one thousand euros, dated on the 31st of December of each year, in the period 2008-2014, expressed in thousand of euros ( $k\text{€}$ ), published by the Spanish Ministry of the Finance and Public Administrations (see [63]).

Table 2 show the main empirical characteristics of the variable of our interest: the number of Spanish indebted municipalities analyzed ( $n$ ); the total amount of borrowing of those indebted municipalities; the debt of the most indebted council; the minimum value of debt considered; the mean and standard deviation (in  $k\text{€}$ ); the skewness and kurtosis of that municipal debt.

**Table 2**

Some relevant information about the datasets considered.

Year	2008	2009	2010	2011	2012	2013	2014
Indebted Municip. ( $n$ )	4,981	5,083	5,039	4,979	5,059	5,028	4,668
Total Amount ( $k\text{€}$ )	$25.2 \times 10^6$	$28.1 \times 10^6$	$28.5 \times 10^6$	$28.2 \times 10^6$	$35.2 \times 10^6$	$34.9 \times 10^6$	$31.3 \times 10^6$
Maximum ( $k\text{€}$ )	$6.7 \times 10^6$	$6.8 \times 10^6$	$6.5 \times 10^6$	$6.3 \times 10^6$	$7.4 \times 10^6$	$7.0 \times 10^6$	$5.9 \times 10^6$
Minimum ( $k\text{€}$ )	1	1	1	1	1	1	1.78
Mean ( $k\text{€}$ )	5,059.2	5,532.2	5,649.1	5,655.6	6,950.6	6,942.4	6,715.4
Std. Dev. ( $k\text{€}$ )	97,714.1	98,603.0	95,643.6	94,555.1	109,764.6	104,657.1	92,645.3
Skewness	64.5	64.1	61.6	61.3	61.7	60.9	57.0
Kurtosis	4,384.5	4,378.4	4,105.7	4,072.0	4,138.9	4,051.3	3,604.9

## 4.2 Power Law behavior in the upper tail

We analyzed the power law behavior of the Spanish municipal debt. For that, we followed the methodology proposed in Clauset et al. [64], based on: (1) the maximum likelihood method, for fitting the Pareto distribution to the data - in this case, the maximum likelihood estimator for the scale parameter is the minimum value of the sample:  $\hat{\sigma} = x_{min}$ , and the maximum likelihood estimator for the shape parameter  $\hat{\alpha}$  is the Hill estimator [65] given by

$$\hat{\alpha} = n \left[ \sum_{i=1}^n \log(x_i/x_{min}) \right]^{-1} ;$$

(2) the Kolmogorov-Smirnov ( $KS$ ) test method based on bootstrap resampling, for testing the goodness-of-fit of Pareto model; and (3), for estimating

the lower bound  $x_{min}$  of the power law behavior, an iterative algorithm where  $x_{min}$  is given by the minimum sample value in which the null hypothesis  $H_0$ : *the data follow a power law model* can't be rejected at 0.1 level of significance.

Table 3 shows, for each year: the shape parameter estimates  $\hat{\alpha}$  obtained from the datasets analyzed; the corresponding scale parameter estimates  $\hat{\sigma}$ , which give us the minimum local debt that follows the power law behavior (in  $k\text{€}$ ); the number of municipalities that follow that behavior; the empirical  $KS$  statistics and the  $p$ -values obtained. It can be seen that power law behavior is only valid in the upper tail of the distribution - only the largest debts can be modeled with a classical Pareto distribution - since null hypothesis  $H_0$ : *the data follow a power law model* can be rejected at the 0.1 level of significance for values of  $x_{min}$  less than  $\hat{\sigma}$  and, in particular, it can be rejected if we considered the whole range of the distribution.

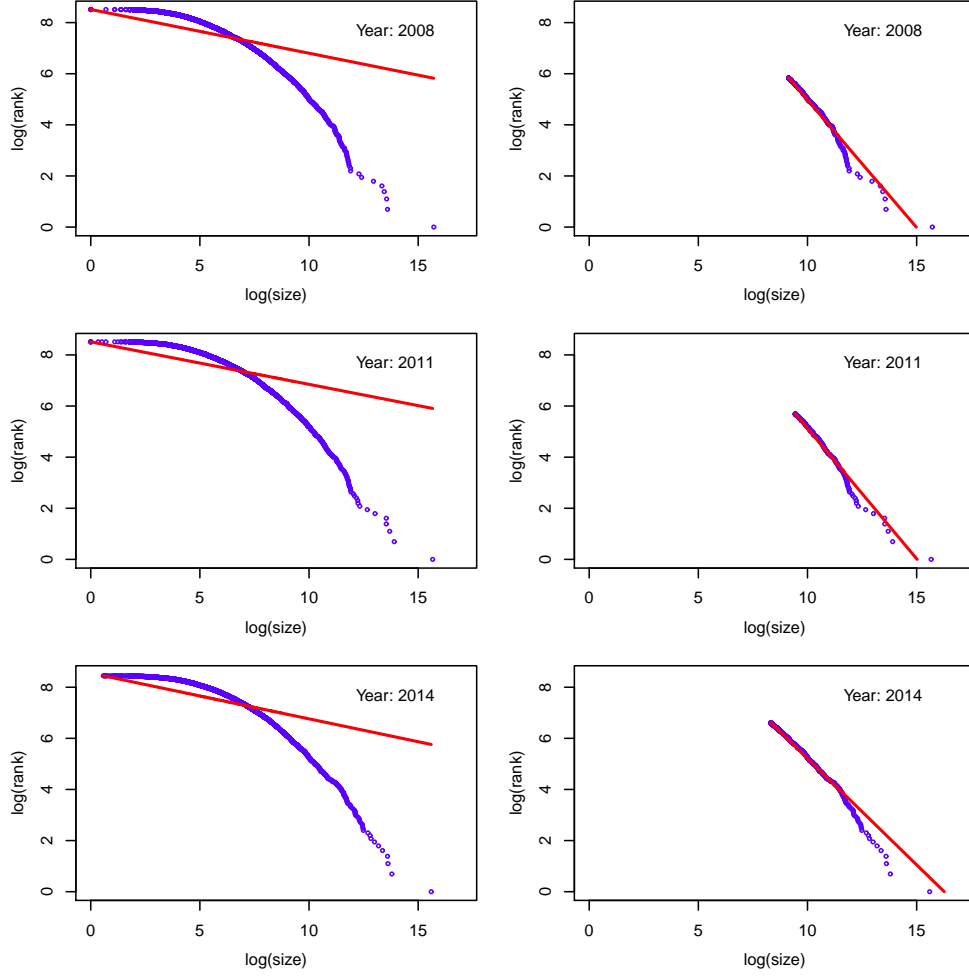
In addition, table 3 shows that shape parameter estimates  $\hat{\alpha}$  are very close to 1 (Zipf's law for many authors, [66, 67, 68, 69, 70, 71, 72, 73, 74]) for the first four years analyzed (2008-2011), and that they change in 2012 (likewise,  $\hat{\sigma}$  and  $n$ ) - coinciding with the political scene change after the Spanish municipal, regional and general elections held on 2011.

**Table 3**

Parameter estimates ( $\hat{\alpha}, \hat{\sigma}$ ) from the Power Law model to the upper tail of the local debt datasets by maximum likelihood; number of municipalities ( $n$ ) with the largest debts, which follow a power law behaviour; empirical  $KS$  statistics; and bootstrap  $p$ -values for that model (values of  $p < 0.1$  indicate that the models can be ruled out with the 0.1 level of significance).

Year	2008	2009	2010	2011	2012	2013	2014
$\hat{\alpha}$ : shape parameter estimates	0.9981	1.0116	0.9990	1.0207	0.8557	0.8455	0.8322
$\hat{\sigma}$ : lower bound ( $k\text{€}$ ) , scale par. estim.	9253	10582	10692	12563	4677	5014	4101
$n$ : size (municipalities) of the upper tail	343	348	345	298	760	700	742
Empirical $KS$ statistics	0.0513	0.0505	0.0505	0.0529	0.0350	0.0364	0.0351
$p$ -value ( $> 0.1$ favor power law model)	0.1037	0.1100	0.1114	0.1293	0.1056	0.1050	0.1084

Figure 4 shows, as a graphical model validation, the rank-size plots (on log-log scale) in the selected years 2008, 2011 and 2014, for the whole range of the datasets (left) and for the upper tail of the distribution (right). Those plots confirm, graphically, that power law model can be ruled out as an adequate model for the whole range of indebted municipalities, and that power law model may serve as an adequate model for municipalities with largest debts above a certain lower bound, in accordance with Table 3.



**Figure 4:** Rank-size plots of the complementary of the cdf multiplied by  $n + 1$  (solid lines) of the classical Pareto distribution (power law model) and the observed data, on log-log scale. Left: Whole range. Right: Upper Tail. Data: Debt of the Spanish indebted municipalities in 2008, 2011 and 2014, whose debt was at least one thousand euros, dated on the 31st of December of each year, in thousand of euros, published by the Spanish Ministry of the Finance and Public Administrations.



### 4.3 The new distribution in the whole range

In this section, we compare the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution with other eight models, and we test the adequacy of the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution to the datasets in the whole range.

We fitted the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  model and those eight known models to the datasets, in the whole range, by maximum likelihood, from 2008 to 2014. Four of those eight models with two parameters: Pareto (Power Law); Lomax (Pareto type II with location parameter  $\mu = 0$ ) [75]; Lognormal [76] and Fisk (Log-logistic) [77] distributions. The other four models with three parameters: Pareto type II; three-parameter lognormal; Burr type XII (Singh-Maddala) [78, 79] and Dagum [80] distributions. Table 4 shows the cumulative distribution functions  $F(x)$  and the probability density functions  $f(x)$  of the nine models considered.

We compared those models using the Bayesian information criterion ( $BIC$ , see Eq.(11)). Table 5 shows the  $BIC$  statistics obtained, from the nine selected models (ranked by  $BIC$ ), corresponding to our datasets in the whole range, from 2008 to 2014.  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution presents the largest values of  $BIC$  statistics, therefore  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution is the model chosen using that model selection criterion. Table 6 shows the corresponding parameter estimates and their standard errors from the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution.

We checked graphically the adequacy of the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution to the datasets in the whole range using rank-size plots. Figure 5 shows the plots obtained from 2008 to 2014.

Finally, we tested the goodness-of-fit of  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution, by a Kolmogorov-Smirnov ( $KS$ ) test method based on bootstrap resampling. Table 7 shows the values of the empirical  $KS$  statistics and the  $p$ -values obtained. It can be seen that we obtained  $p$ -values  $\geq 0.1$  in three of the seven years considered.

In summary,  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution can be useful for modeling Spanish municipalities debt: it presents the best  $BIC$  statistics of the nine selected models; graphically it gives a reasonable description of the datasets; and it cannot be rejected with 0.1 level of significance in three of the seven years considered.

**Table 4**

Cumulative distribution functions and probability density functions of the models fitted to the dataset in the whole range.  $\Phi(z)$  denotes the standard normal CDF.

Distribution	$F(x)$	$f(x)$
Pareto	$1 - \left(\frac{x}{\sigma}\right)^{-\alpha}$	$\frac{\alpha\sigma^\alpha}{x^{\alpha+1}} \quad x \geq \sigma$
Lomax	$1 - \left(1 + \frac{x}{\sigma}\right)^{-\alpha}$	$\frac{\alpha\sigma^\alpha}{(x+\sigma)^{\alpha+1}} \quad x \geq 0$
Lognormal	$\Phi\left(\frac{\log x - \mu}{\sigma}\right)$	$\frac{1}{x\sigma\sqrt{2\pi}} \exp\left[-\frac{(\log x - \mu)^2}{2\sigma^2}\right] \quad x > 0$
Fisk	$\frac{1}{1 + (x/\alpha)^{-\beta}}$	$\frac{(\beta/\alpha)(x/\alpha)^{\beta-1}}{(1 + (x/\alpha)^\beta)^2} \quad x > 0$
Pareto II	$1 - \left(1 + \frac{x - \mu}{\sigma}\right)^{-\alpha}$	$\frac{\alpha\sigma^\alpha}{(x - \mu + \sigma)^{\alpha+1}} \quad x \geq \mu$
Lognormal 3p	$\Phi\left(\frac{\log(x - \gamma) - \mu}{\sigma}\right)$	$\frac{1}{\sigma(x - \gamma)\sqrt{2\pi}} \exp\left[-\frac{(\log(x - \gamma) - \mu)^2}{2\sigma^2}\right] \quad x > \gamma$
Burr type XII	$1 - \left[1 + \left(\frac{x}{b}\right)^a\right]^{-q}$	$\frac{aqx^{a-1}}{b^a[1 + (x/b)^a]^{q+1}}, \quad x \geq 0$
Dagum	$\left[1 + \left(\frac{x}{b}\right)^{-a}\right]^{-p}$	$\frac{apx^{a-1}}{b^a[1 + (x/b)^a]^{p+1}}, \quad x \geq 0$
$\mathcal{GPL}(\alpha, \beta, \sigma)$	$1 - \exp\left\{-\alpha \frac{[\log(x/\sigma)]^{\beta+1}}{[\log(x/\sigma) + 1]^\beta}\right\}$	$\frac{\alpha}{x} \left[\frac{\log(x/\sigma) + 1 + \beta}{\log(x/\sigma) + 1}\right] \left[\frac{\log(x/\sigma)}{\log(x/\sigma) + 1}\right]^\beta \exp\left\{-\alpha \frac{[\log(x/\sigma)]^{\beta+1}}{[\log(x/\sigma) + 1]^\beta}\right\}$

**Table 5**

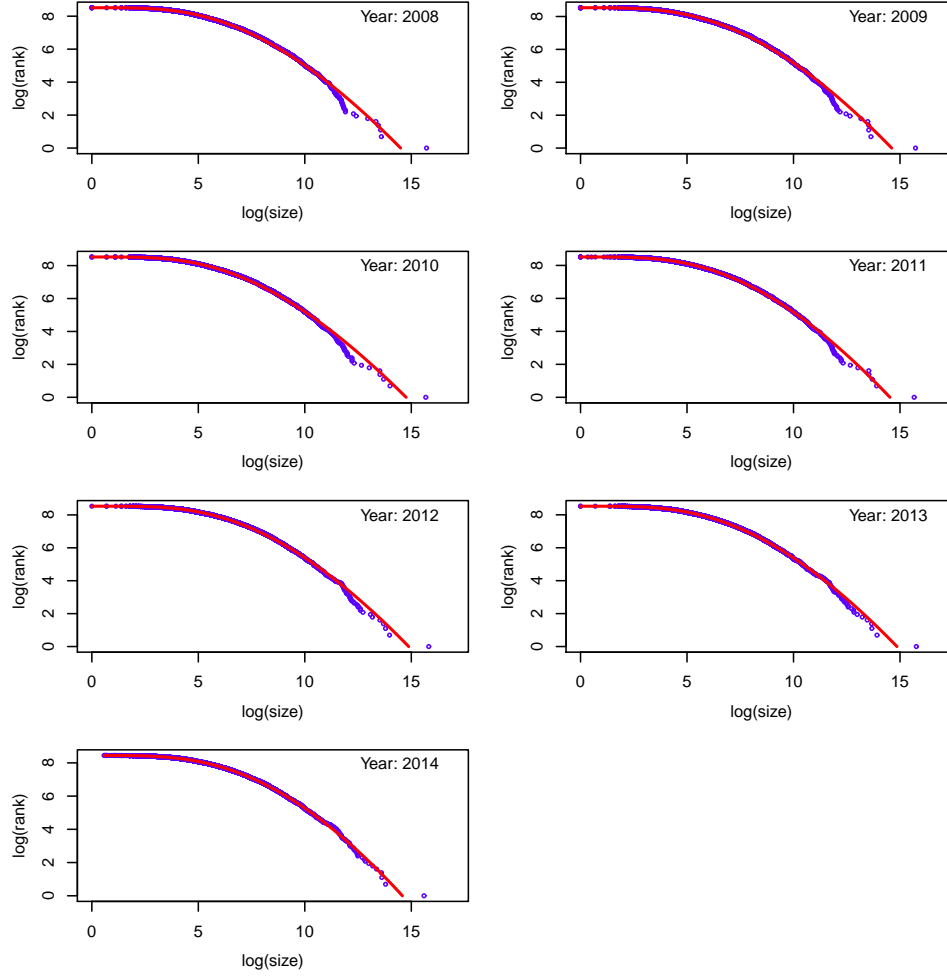
*BIC* statistics for nine candidate models, fitted by maximum likelihood to municipal debt data in Spain. Larger values indicate better fitted models (models appear ranked by *BIC*).

Year	2008	2009	2010	2011	2012	2013	2014
$\mathcal{GPL}(\alpha, \beta, \sigma)$	-39656	-41041	-40892	-40528	-42316	-41925	-38840
Lognormal 3p	-39723	-41092	-40954	-40579	-42368	-41974	-38859
Dagum	-39708	-41093	-40955	-40578	-42380	-41985	-38884
Lognormal	-39733	-41101	-40966	-40587	-42378	-41984	-38877
Pareto II	-39731	-41119	-40986	-40607	-42417	-42015	-38893
Lomax	-39748	-41135	-41000	-40620	-42427	-42026	-38914
Burr type XII	-39746	-41139	-41003	-40624	-42432	-42030	-38917
Fisk	-39792	-41172	-41036	-40652	-42448	-42046	-38927
Pareto	-42870	-44251	-44200	-43820	-45819	-45338	-41486

**Table 6**

Parameter estimates from the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  model, to Spanish municipal debt datasets, in the whole range, by maximum likelihood (standard errors in parenthesis).

Year	2008	2009	2010	2011	2012	2013	2014
$\hat{\alpha}$	2.3477 (0.1910)	2.6126 (0.2336)	2.3085 (0.1979)	2.6703 (0.2419)	2.5098 (0.2387)	2.5722 (0.2445)	3.0427 (0.3325)
$\hat{\beta}$	24.3346 (1.3320)	27.4832 (1.5758)	24.5065 (1.4319)	27.1347 (1.5735)	26.1137 (1.6499)	26.9485 (1.6755)	30.5616 (2.0480)
$\hat{\sigma}$	0.2367 (0.0392)	0.1536 (0.0291)	0.2536 (0.0454)	0.1882 (0.0355)	0.2625 (0.0530)	0.2199 (0.0447)	0.1335 (0.0316)



**Figure 5:** Rank-size plots of the complementary of the cdf multiplied by  $n + 1$  (solid lines) of the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution and the observed data, on log-log scale. Data: Debt of the Spanish indebted municipalities, from 2008 to 2014, whose debt was at least one thousand euros, dated on the 31st of December of each year, in thousand of euros, published by the Spanish Ministry of the Finance and Public Administrations.

**Table 7**

Empirical KS statistics and bootstrap  $p$ -values for  $\mathcal{GPL}(\alpha, \beta, \sigma)$  model ( $p \geq 0.1$  favor GPL model)

Year	2008	2009	2010	2011	2012	2013	2014
$KS$	0.0180	0.0181	0.0139	0.0144	0.0144	0.0098	0.0093
$p$ -value	0.0000	0.0000	0.1000	0.0048	0.0056	0.1977	0.3258

## 5 Conclusions

In this study, we focussed on modeling the whole range of empirical data, whose upper tail follows a power-law behaviour. To do that, we modeled the exponent of the classical Pareto distribution using a non-linear function.

We found a new family of Generalized Power Law distributions, with three parameters, which includes Pareto (power law, Pareto type I) and PPS distributions as special cases. We showed that it is a genuine family of distributions. We provided some particular functional forms of that family. And, as an extension, we presented two more new families of distributions based on Pareto type II and Pareto type IV distributions respectively.

We found a new distribution, the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution, which belongs to the new family of Generalized Power Law distributions described previously. We showed that  $\mathcal{GPL}(\alpha, \beta, \sigma)$  and Power Law models are right tail equivalent ( $\mathcal{GPL}(\alpha, \beta, \sigma)$  model exhibit a power law behavior in the tail) and that  $\mathcal{GPL}(\alpha, \beta, \sigma)$  model belongs to the Maximum Domain of Attraction of the Fréchet distribution, which means that  $\mathcal{GPL}(\alpha, \beta, \sigma)$  is a heavy-tailed distribution and it can be useful for statistical modeling of real phenomena with extremely large observations. We provided the genesis, the basic properties (including the quantile function for computer simulation), and the corresponding estimation and testing methods for that distribution.

Finally, we provided empirical evidence of the efficacy of the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution (and for extension, of the new family of Generalized Power Law distribution) with real datasets in the whole range. In particular, we showed that  $\mathcal{GPL}(\alpha, \beta, \sigma)$  model can be useful for modeling municipal debt data. For that, we considered information of Spanish indebted municipalities, whose debt was at least one thousand euros, in the period 2008-2014, published by the Spanish Ministry of the Finance and Public Administrations. We showed that the Spanish municipal debt follows a power law in the tail but not in the whole range. And finally, we showed analytically and graphically the competence of the  $\mathcal{GPL}(\alpha, \beta, \sigma)$  distribution with municipal debt data

in the whole range, in comparison with other known distributions as the Lognormal, the Generalized Pareto, the Fisk, the Burr type XII and the Dagum distributions.

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